

Vorticity and level-set variations of invariant current bound steady-state dissipation

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Abstract

A non-vanishing entropy production rate is a hallmark of non-equilibrium stationary states and is therefore at the heart of non-equilibrium thermodynamics. It is a manifestation of a steady circulation \mathbf{J}_{inv} along the level sets of the invariant density ρ_{inv} , and is thus generically used to quantify how far a steady system is driven out of equilibrium. While it is well known that there exists a continuum of distinct steady states with the same invariant measure, the question how the geometry and topology of the invariant current *a priori* affect dissipation remained elusive. For confined irreversible diffusions we identify two minimal descriptors, the ρ_{inv} -weighted vorticity and the variation of \mathbf{J}_{inv} along level sets of ρ_{inv} , and prove that these jointly bound from above the steady-state entropy production rate. In regions where ρ_{inv} is close to Gaussian the bound is dominated solely by the vorticity of the drift field and in the low-noise (Freidlin–Wentzel) limit by any non-potential contribution to the drift, rendering \mathbf{J}_{inv} virtually constant along the level sets of ρ_{inv} .

Keywords: stochastic thermodynamics, entropy production, irreversible diffusion, vorticity, Sobolev inequalities

1 Introduction

A defining feature of non-equilibrium as opposed to equilibrium steady states is a positive entropy production [1–8]. The thermodynamics of far-from-equilibrium systems is essential for elucidating the physical principles that sustain driven, in particular living matter [9–13]. Erwin Schrödinger observed already in 1944 [14] that living organisms must continuously increase the entropy of its surroundings in order to avoid decaying to thermodynamic equilibrium. Far-from-equilibrium thermodynamics was put on formal grounds by the “Brussels school” [15] and was more recently extended to Stochastic Thermodynamics [16] focussing on a sample-path perspective.

Stochastic Thermodynamics provides a quite general framework that, in principle, applies arbitrarily far from equilibrium [16]. The main constraining assumption of Stochastic Thermodynamics is that of *local equilibrium*, stating that all unobserved degrees of freedom evolve sufficiently fast with respect to the observed ones, such that they may be faithfully considered to be, at every instance, at thermodynamic equilibrium (i.e. “Gibbsian”) at the temperature of the surroundings (also called “the heat bath”) [1, 17–23]. Within this local-equilibrium paradigm the total entropy flow along an individual *stochastic* sample path ω compares the probability measure of a path, $P_t(\omega)$, with the probability measure of the respective time-reversed path, $P_{\theta t}(\omega)$, both defined on the σ -algebra \mathcal{F}_τ^t generated by the process $(x_\tau)_{0 \leq \tau \leq t}$, and is defined as

$$\Sigma_t(\omega) \equiv \ln \frac{dP_t}{dP_{\theta t}}(\omega)$$

where $\theta(x_\tau)_{0 \leq \tau \leq t} \equiv \epsilon(x_{t-\tau})_{0 \leq \tau \leq t}$ denotes the time-reversal operation and, properly defined for the given setting; in particular, ϵ reverses the sign of all degrees of freedom with odd time parity (see e.g. [1, 17–23] for various settings). In typical physically relevant settings [1, 22], and in particular in the case of overdamped diffusions considered herein (see [24]) $P(\omega)$ may be assumed to possess a density $p(x_\tau(\omega))$ and $\Sigma(\omega)$ is defined via the log-ratio of the respective probability densities $p(x_\tau(\omega))$ and $p(\theta x_\tau(\omega))$.

In this work we focus exclusively on strongly ergodic dynamics with invariant measure P_{inv} that also possesses a density ρ_{inv} and, as a result of broken detailed balance, carries a stationary invariant current J_{inv} . In this setting the thermodynamic observable of central interest is the *steady state entropy production rate* (or steady-state dissipation) defined as

$$\dot{\Sigma}_s \equiv \lim_{t \rightarrow \infty} \frac{1}{t} \ln \frac{dP_t}{dP_{\theta t}}(\omega) = \lim_{\Delta t \rightarrow 0^+} \frac{1}{\Delta t} \mathbb{E}_{\Delta t}^{\text{inv}} \left(\ln \frac{p([x_\tau]_{0 \leq \tau \leq \Delta t})}{p([\theta x_\tau]_{0 \leq \tau \leq \Delta t})} \right), \quad (1)$$

which characterizes the degree of irreversibility and hence non-equilibrium and where the second equality follows from x_τ being ergodic. $\mathbb{E}_t^{\text{inv}}(\cdot)$ denotes the expectation over $p([x_\tau]_{0 \leq \tau \leq t})$ in the steady-state ensemble, i.e. over paths

$(x_\tau)_{0 \leq \tau \leq t}$ propagating from the initial density $p(x_0) = \rho_{\text{inv}}(x_0)$ up to time t . The limits in Eq. (1) exist [24].

We consider overdamped diffusions evolving according to the stochastic differential equation [25]

$$dx_\tau = F(x_\tau)d\tau + \sqrt{2D}dW_\tau \quad (2)$$

with $x_\tau \in \mathbb{R}^d$, a smooth confining drift field $F: \mathbb{R}^d \rightarrow \mathbb{R}^d$, the $d \times d$, symmetric positive definite matrix D , and W_τ denoting the d -dimensional Wiener process. F is supposed to have nice growth properties and to locally satisfy Lipschitz conditions guaranteeing the existence and uniqueness of a solution of Eq. (2) [25]. The square root of \sqrt{D} is defined as usual as $S^T \sqrt{\lambda_i} S$ via orthogonal diagonalization $SDS^T = \text{diag}(\lambda_i)$ with $S^T = S^{-1}$.

From a physical perspective Eq. (2) embodies stochastic evolutions in low Reynolds media such as macromolecules [26], molecular machines [27], or suspensions of colloidal particles [28].

The Ito equation (2) translates to the Fokker-Planck (or forward Kolmogorov) equation for the probability density $P: \mathbb{R}^d \times [0, \infty) \rightarrow \mathbb{R}^+$

$$\partial_t P(x, t) = \nabla \cdot [D\nabla - F(x)]P(x, t) \equiv -\nabla \cdot J(x, t) \quad (3)$$

where ∇ is the d -dimensional gradient operator and we defined the current density $J(x, t) \equiv (F(x) - D\nabla)P(x, t)$, $J: \mathbb{R}^d \times [0, \infty) \rightarrow \mathbb{R}^d$. By our assumptions we have $\lim_{t \rightarrow \infty} P(x, t) = \rho_{\text{inv}}(x)$ and $\lim_{t \rightarrow \infty} J(x, t) = J_{\text{inv}}(x)$ and obviously $\nabla \cdot J_{\text{inv}}(x) = 0$, i.e. the invariant current is incompressible.

As a result of the incompressibility of the stationary current, the drift F of the ergodic diffusion in Eq. (2) may be decomposed into reversible F_{rev} and irreversible F_{irr} [8, 29] (alternatively called “conservative” and “dissipative” [30, 31]) components, $F(x) = F_{\text{rev}}(x) + F_{\text{irr}}(x)$ defined as

$$\begin{aligned} F_{\text{rev}}(x) &\equiv D\nabla \ln \rho_{\text{inv}}(x) \equiv -D\nabla \Phi(x) \\ F_{\text{irr}}(x) &\equiv \frac{J_{\text{inv}}(x)}{\rho_{\text{inv}}(x)} = A(x)\nabla \ln \rho_{\text{inv}}(x) + \nabla \cdot A(x), \end{aligned} \quad (4)$$

where $\Phi(x)$ is the generalized “nonequilibrium” potential, $A(x)$ is a $d \times d$ anti-symmetric matrix ($A(x)^T = -A(x)$) and we define the divergence of a matrix as $(\nabla \cdot A(x))_k \equiv \sum_{l=1}^d \partial_{x_l} A_{kl}(x)$. One can henceforth show (see e.g. [16, 24]) that

$$\dot{\Sigma}_s = \int dx F_{\text{irr}}(x) \cdot D^{-1} F_{\text{irr}}(x) \rho_{\text{inv}}(x) \geq 0 \quad (5)$$

The intuition about $A(x)$ in Eq. (4) is that it describes variations of the $J_{\text{inv}}(x)$ along the level sets of $\rho_{\text{inv}}(x)$. That is, if $A(x)$ is a constant matrix (i.e. $\nabla \cdot A = 0$) then $|J_{\text{inv}}(x)|$ is constant on level sets of $\rho_{\text{inv}}(x)$ and $F_{\text{rev}}(x) \perp F_{\text{irr}}(x)$, $\forall x$. However, when $A(x)$ is not constant (which is typically the case), $F_{\text{irr}}(x)$ has a component in the direction of $F_{\text{rev}}(x)$, contained in $\nabla \cdot A(x)$, and it is generally

impossible to determine $A(x)$ from $F(x)$, and thus steady-state dissipation $\dot{\Sigma}_s$ via Eq. (5), without first determining $\rho_{\text{inv}}(x)$, and there is no general way to do so. Note that a Helmholtz-Hodge type may deliver mutually orthogonal fields, but these will not be $F_{\text{rev}}(x)$ and $F_{\text{irr}}(x)$. Therefore, seemingly not as much can be inferred about the nonequilibrium thermodynamics of the process (2) directly from $F(x)$ alone. Moreover, even in cases when both $\rho_{\text{inv}}(x)$ and $F_{\text{irr}}(x)$ are known, an understanding of how the properties of $\rho_{\text{inv}}(x)$ and $F_{\text{irr}}(x)$, and in particular the topology of $J_{\text{inv}}(x)$, affect $\dot{\Sigma}_s$ remains elusive.

Notwithstanding its physical importance, $\dot{\Sigma}_s$ is very difficult to quantify in general, as it requires detailed knowledge about *all* dissipative degrees of freedom, i.e. the knowledge of $\rho_{\text{inv}}(x)$ and $J_{\text{inv}}(x)$. Important insight can be obtained in the form of experimentally accessible lower bounds via the so-called Thermodynamic Uncertainty Relation [5, 32–38], which bounds $\dot{\Sigma}_s$ from below by means of any time-accumulated current observable inferred from individual stochastic trajectories. This approach indeed provides a very powerful means of inferring $\dot{\Sigma}_s$, but does not give insight into what properties of $F(x)$ relate to $\dot{\Sigma}_s$ and in what manner. For this, one needs a more direct connection between the two that does not require the knowledge of $\rho_{\text{inv}}(x)$ as an input.

An important rigorous result in this direction was obtained in terms of the *circulation decomposition*, the sum of an “observed” and “hidden circulation” around some closed curve and note that both contributions are positive [24] (see also a more recent work on a related topic [39]). This approach, notably, does not require to know $\rho_{\text{inv}}(x)$ nor $J_{\text{inv}}(x)$. However, it depends explicitly on the choice of the closed curve and therefore can in practice serve as a lower bound on $\dot{\Sigma}_s$, as the possibility of missing the “hidden” circulation for a selected closed curve cannot be dismissed if one does not know the topology of $J_{\text{inv}}(x)$.

There is thus a need for results that directly relate the properties of $F(x)$ to $\dot{\Sigma}_s$ in a general setting. Here we make progress in this direction by proving upper bounds on $\dot{\Sigma}_s$ in terms of the vorticity of $F(x)$ and variation of $J_{\text{inv}}(x)$ on level sets of $\rho_{\text{inv}}(x)$ by using Sobolev inequalities.

In the (low-noise) Freidlin-Wentzel limit, i.e. $\|D\| \rightarrow 0$ we show that the bound on $\dot{\Sigma}_s$ becomes dominated by $\nabla\Phi(x) \times F_{\text{irr}}(x)$ alone, which excludes any contributions parallel to $\nabla\Phi(x)$, and in fact if $F_{\text{rev}}(x)$ is parallel to $\nabla\Phi(x)$ then $F_{\text{irr}}(x)$ may be replaced by $F(x)$. Moreover, in regions where $\rho_{\text{inv}}(x)$ is essentially a Gaussian (or can be well approximated by one) the upper bound on $\dot{\Sigma}_s$ is as well dominated solely by vorticity. Notably, as forward and backward paths between pairs of points are nominally expected to be different in vortex flow, the fact that vorticity bounds time-irreversibility from above seems quite intuitive. Our results provide a direct and intuitive link between the topology of $J_{\text{inv}}(x)$ and the steady-state dissipation $\dot{\Sigma}_s$. By bounding $\dot{\Sigma}_s$ from above, they are complementary to the lower bounds established by the Thermodynamic Uncertainty relation. The results in the low-noise limit, moreover, provide further support for the so-called “local detailed balance” structure of

transition rates assumed in constructing approximate Markov-jump representations of dynamic on long time-scales.

The paper is structured as follows. In Sec. 2 we present a concise summary of our main results and discuss their physical implications. In Sec. 3 we then give the precise definitions and state the essential preparatory theorems. In Sec. 4 we present the detailed proofs of the results.

2 Summary of the main results

We throughout assume that $x_\tau \in \Omega$ and that the domain $\Omega \subset \mathbb{R}^2$ or \mathbb{R}^3 is bounded, Lipschitz-continuous and simply-connected. We seek insight into which properties of $J_{\text{inv}}(x)$ and in turn $F_{\text{irr}}(x)$ of the overdamped diffusion in Eq. (2) determine the steady-state dissipation $\dot{\Sigma}$ defined in Eq. (5).

We will show that these turn out to be the ρ_{inv} -weighted *vorticity*

$$V^q \equiv \int_{\Omega} |\nabla \times F_{\text{irr}}(x)|^{2q} \rho_{\text{inv}}(x) dx,$$

where $q \geq 1$ and with $V^1 \equiv V$, the ρ_{inv} -weighted level-set variations of $J_{\text{inv}}(x)$

$$\delta_{\text{LS}}^q \equiv \int_{\Omega} |\nabla \cdot F_{\text{irr}}|^{2q} \rho_{\text{inv}}(x) dx = \int_{\Omega} |\nabla \Phi(x) \cdot F_{\text{irr}}(x)|^{2q} \rho_{\text{inv}}(x) dx,$$

with $\delta_{\text{LS}}^1 \equiv \delta_{\text{LS}}$, and the ρ_{inv} -weighted deviations of $J_{\text{inv}}(x)$ from alignment with $\nabla \Phi(x)$

$$\delta_{\nabla \Phi}^q \equiv \int_{\Omega} |\nabla \Phi(x) \times F_{\text{irr}}(x)|^{2q} \rho_{\text{inv}}(x) dx,$$

with $\delta_{\nabla \Phi}^1 \equiv \delta_{\nabla \Phi}$.

The vorticity V^q is defined in a standard fashion and the ρ_{inv} weight is intended to “grade” the local vorticity features according to their physical relevance in the steady-state ensemble. When $D = \text{const} \cdot \mathbb{1}$ we have $\nabla \times D \nabla \Phi(x) = 0$ and we may replace $\nabla \times F_{\text{irr}}(x) \rightarrow \nabla \times F(x)$ everywhere.

To provide intuition about why δ_{LS}^q measures level-set variations of $J_{\text{inv}}(x)$ we note that if $J_{\text{inv}}(x)$ flows perfectly along the level sets of $\rho_{\text{inv}}(x)$ and the level sets are closed curves, then the conservation of mass requires $|J_{\text{inv}}(x)|$ must be constant along each level set. As $\rho_{\text{inv}}(x)$ is by definition constant on level sets, $J_{\text{inv}}(x)$ being mass-preserving is equivalent to $F_{\text{irr}}(x)$ being volume-preserving i.e. divergence free. Hence, any level-set variation of $|J_{\text{inv}}(x)|$ leads to non-zero $\nabla \cdot F_{\text{irr}}(x)$ and non-zero $\nabla \Phi(x) \cdot F_{\text{irr}}(x)$.

Just as δ_{LS}^q essentially measures deviations of $J_{\text{inv}}(x)$ from being perfectly orthogonal to $\nabla \Phi(x)$ on level sets of ρ_{inv} , the quantity $\delta_{\nabla \Phi}^q$ measures how much $J_{\text{inv}}(x)$ deviates from being parallel to the potential gradient $\nabla \Phi(x)$.

Under these assumptions (that will be detailed further below) we prove the following upper bounds.

6 Upper bounds on dissipation in irreversible diffusions

1. For a general diffusion (2) in nonequilibrium steady state we find (see Theorem 6)

$$\dot{\Sigma}_s \leq C(\Omega) \left[V + \frac{1}{4}(\delta_{\text{LS}} + \delta_{\nabla\Phi}) \right],$$

where $C(\Omega)$ is a constant depending only of Ω .

2. In the Freidlin-Wentzel limit when $\|D\| = \mathcal{O}(\epsilon) \rightarrow 0$ in Eq. (2), we prove (see Corollary 7) that to leading order in ϵ

$$\dot{\Sigma}_s^\epsilon \leq C(\Omega) \int_{\Omega} |\nabla\Phi^\epsilon \times F_{\text{irr}}^\epsilon(x)|^2 \rho_{\text{inv}}^\epsilon dx,$$

where $C(\Omega)$ is a constant depending only on the domain Ω . Moreover, if the reversible drift is parallel to the potential gradient, we may replace $F_{\text{irr}}^\epsilon(x)$ by *full* drift field $F(x)$, suggesting that as $\|D\| \rightarrow 0$ (i.e. in the low-noise limit) dissipation is bounded from above directly by any non-potential contribution to F in regions where $\rho_{\text{inv}}^\epsilon$ has substantial mass.

3. When $\sup_{\Omega} \rho_{\text{inv}} < \infty$ and $\rho_{\text{inv}}^{1/(1-q)} \in L^1(\Omega)$ for $1 < q < \infty$ we show (see Theorem 8)

$$\dot{\Sigma}_s \leq C(\Omega, \rho_{\text{inv}}) \left[(V^q)^{1/q} + (\delta_{\text{LS}}^q)^{1/q} \right]$$

where

$$C(\Omega, \rho_{\text{inv}}) = C(\Omega) \cdot \sup_{\Omega} \rho_{\text{inv}} \cdot \left(\int_{\Omega} \rho_{\text{inv}}^{1/(1-q)} dx \right)^{(q-1)/q}.$$

is a constant depending only on Ω and ρ_{inv} .

4. Moreover, if in addition $\inf_{\Omega} \rho_{\text{inv}} > 0$ we have that (see Corollary 9)

$$\dot{\Sigma}_s \leq C(\Omega) \sup_{\Omega} \rho_{\text{inv}} \left(\inf_{\Omega} \rho_{\text{inv}} \right)^{-1} [V + \delta_{\text{LS}}],$$

where the constant $C(\Omega)$ depends only on Ω .

5. If on a subdomain $\omega \subset \Omega$ the generalized potential $\Phi(x)$ is essentially a parabola with principal axes $K_i > 0$ and the diameter of ω is smaller than

$(\sum_{i=1}^d K_i)^{-1/2}$, it holds that

$$\int_{\omega} |F_{\text{irr}} \cdot \nabla \Phi|^2 dx \lesssim \left(\sum_{i=1}^d K_i \right) \text{diam}(\omega)^2 \int_{\omega} |\nabla \times F_{\text{irr}}|^2 dx,$$

where in $d = 3$ the above result is subject to some symmetry requirements. The above result suggests that in the vicinity of local minima of $\Phi(x)$ level-set variations of $J_{\text{inv}}(x)$ are sub-dominant and we may essentially regard $J_{\text{inv}}(x)$ as being orthogonal to $\nabla \Phi(x)$ on every subdomain ω . This seems to be particularly important in the $\|D\| \rightarrow 0$ limit.

The fact that the upper bound on $\dot{\Sigma}_s$ depends on the vorticity of F_{irr} is expected, since vortices nominally break the forward-backward symmetry of sample paths between two given points. However, the remaining two contributions seem less obvious. That is, $\dot{\Sigma}_s$ essentially reflects the energetic cost of maintaining the nonequilibrium steady state and thus it *a priori* does not seem to be obvious that F_{irr} with a component in the direction of $\nabla \Phi(x)$ should “cost” more.

The seemingly most insightful result is that in the Freidlin-Wentzel limit ($\|D\| \rightarrow 0$), which physically would correspond to a $\Phi(x)$ with deep minima separated by high barriers. Here non-vanishing $F_{\text{irr}}(x)$ imply that transitions between two minima in opposite directions may occur along distinct *instanton* pathways (see e.g. [40, 41]). These instantons are minimum-action paths between basins around which the reactive sample paths x_{τ} concentrate tightly as $\|D\| \rightarrow 0$ [40, 41]. On long time-scales, longer than the typical escape time over a typical barrier of $\Phi(x)$ which is exponential in the barrier height [40–42], such dynamics has an effective representation with a Markov jump process on a state space $\{i\}$ [43] which is typically assumed to obey “local detailed balance” [44, 45], i.e. with transition rates between states i and j given by

$$k_{i \rightarrow j} / k_{j \rightarrow i} = e^{\phi(i) - \phi(j) - A_{ij}} \quad (6)$$

with antisymmetric link “affinity” $A_{ij} = -A_{ji}$. Note that here ϕ_i include, in addition to $\Phi(x)$, the entropy of the basin [44, 45]. On such a reduced state space $\dot{\Sigma}_s$ and hence time-irreversibility is determined solely by the affinities $A_{ji} \neq 0$ [16]. In full generality the problem of determining $k_{i \rightarrow j}$ for an irreversible diffusive dynamics (2) in the $\|D\| \rightarrow 0$ limit remains unsolved. Important progress has been made under the assumption of a *transverse decomposition* [41] which in our setting corresponds to the simplifying assumption that A in Eq. (4) is a constant matrix and hence $F_{\text{irr}}(x) \perp F_{\text{rev}}(x)$ everywhere. Under these assumptions, the main result of [41] (see Eq. (1.10) in conjunction with Eq. (2.12) therein) indeed implies the structure of Eq. (6).

The asymptotic correspondence seems to be reflected in our results; i.e. according to Corollary 7 in the limit $\|D\| \rightarrow 0$ the affinities A_{ij} in Eq. (6) should indeed arise from local contributions to $\nabla \Phi^{\epsilon}(x) \times F_{\text{irr}}^{\epsilon}(x)$ that exclude any contributions parallel to $\nabla \Phi^{\epsilon}(x)$ along the distinct forward-backward

minimum-action (instanton) transition paths between basins. This seems to agree with the notion of jump rates obeying local detailed balance insofar as our upper bounds are reasonably sharp.

3 Setup

Let Ω be an open subset of \mathbb{R}^d with boundary Γ , along which n denotes the unit normal vector. Consider the overdamped diffusion process in Eq. (2) on Ω with steady state density $\rho_{\text{inv}}(x)$, which is the unique normalised solution of the boundary value problem

$$\begin{cases} \nabla \cdot [D\nabla\rho_{\text{inv}}(x) - F(x)\rho_{\text{inv}}(x)] = 0 & \text{in } \Omega \\ [D\nabla\rho_{\text{inv}}(x) - F(x)\rho_{\text{inv}}(x)] \cdot n = 0 & \text{on } \Gamma. \end{cases}$$

In the following we define the main quantities of interest.

Definition 1 The invariant current is defined as

$$J_{\text{inv}}(x) \equiv F(x)\rho_{\text{inv}}(x) - D\nabla\rho_{\text{inv}}(x) = F_{\text{irr}}(x)\rho_{\text{inv}}(x),$$

the “non-equilibrium” potential as

$$\Phi(x) \equiv -\ln\rho_{\text{inv}}(x),$$

where from follows the reversible part of $F(x)$, defined as

$$F_{\text{rev}}(x) \equiv -D\nabla\Phi(x),$$

while the irreversible part of $F(x)$ is defined as

$$F_{\text{irr}}(x) \equiv F(x) + D\nabla\Phi(x) = -A(x)\nabla\Phi + \nabla \cdot A(x).$$

The ρ_{inv} -weighted vorticity of the drift field $F(x)$ is defined as

$$V \equiv \int_{\Omega} |\nabla \times F_{\text{irr}}(x)|^2 \rho_{\text{inv}}(x) dx \stackrel{D=\text{const}\mathbf{1}}{=} \int_{\Omega} |\nabla \times F(x)|^2 \rho_{\text{inv}}(x) dx,$$

where the last equality holds provided that D is isotropic i.e. a constant multiple of the identity matrix.

To demonstrate our results, we need to further introduce some preliminaries on Sobolev inequalities. We will now state the theorems to be applied in the derivation of our results and sketch some proofs, while for the technical details we refer to [46] (Chapter I \mathbf{S}_1 - \mathbf{S}_3).

We will throughout assume that Ω is a bounded, simply-connected open subset of \mathbb{R}^d with $d = 2$ or 3 , and the boundary $\Gamma = \bigcup_{i=0}^p \Gamma_i$ is Lipschitz-continuous. Recall that a region is simply connected if every closed curve within it can be continuously deformed to a single point in that region. In \mathbb{R}^2 , a simply connect region has no holes and the boundary Γ consists of only one piece. In \mathbb{R}^3 , we can allow for holes in the middle without going all the way through the region, and we can have $p \geq 1$. The duality between $H^{-1/2}(\Gamma_i)$ and $H^{1/2}(\Gamma_i)$ will be denoted by $\langle \cdot, \cdot \rangle_{\Gamma_i}$, and (\cdot, \cdot) will be used to represent the scalar product

on $L^2(\Omega)^d$. We will present the results for dimension $d = 3$, the same results also hold for dimension $d = 2$ for which we refer to e.g. [46] (Proposition 3.1) for details.

Let $v : \Omega \rightarrow \mathbb{R}^d$ be a vector field. We set up the notations for the following L^2 -based Sobolev spaces:

$$H(\operatorname{div}; \Omega) \equiv \{v \in L^2(\Omega)^d : \nabla \cdot v \in L^2(\Omega)\}$$

which is a Hilbert space equipped with the norm

$$\|v\|_{H(\operatorname{div}; \Omega)} = [\|v\|_{L^2(\Omega)}^2 + \|\nabla \cdot v\|_{L^2(\Omega)}^2]^{1/2}.$$

We define the subspace of vector fields with zero flux across Γ :

$$H_0(\operatorname{div}; \Omega) \equiv \{u \in H(\operatorname{div}; \Omega) : u \cdot n|_{\Gamma} = 0\},$$

as well as the subspace with both zero flux across Γ and zero divergence in Ω :

$$H \equiv \{u \in L^2(\Omega)^d : u \cdot n|_{\Gamma} = 0, \nabla \cdot u = 0\}.$$

As for the space $H(\operatorname{div}; \Omega)$, we introduce the space $H(\operatorname{curl}; \Omega)$:

$$H(\operatorname{curl}; \Omega) \equiv \{v \in L^2(\Omega)^d; \nabla \times v \in L^2(\Omega)^d\},$$

which is a Hilbert space equipped with the norm

$$\|v\|_{H(\operatorname{curl}; \Omega)} = [\|v\|_{L^2(\Omega)}^2 + \|\nabla \times v\|_{L^2(\Omega)}^2]^{1/2}.$$

Finally, we define

$$\begin{aligned} U &\equiv H_0(\operatorname{div}; \Omega) \cap H(\operatorname{curl}; \Omega) \\ &= \{\phi \in L^2(\Omega)^d : \nabla \cdot \phi \in L^2(\Omega), \nabla \times \phi \in L^2(\Omega)^d, \phi \cdot n|_{\Gamma} = 0\} \end{aligned}$$

and the divergence-free subspace

$$\begin{aligned} U_0 &\equiv \{\phi \in U; \nabla \cdot \phi = 0\} \\ &= \{\phi \in L^2(\Omega)^d : \nabla \cdot \phi = 0, \nabla \times \phi \in L^2(\Omega)^d, \phi \cdot n|_{\Gamma} = 0\}. \end{aligned}$$

U is a Hilbert space equipped with the norm

$$\|\phi\| = \|\phi\|_{L^2(\Omega)} + \|\nabla \cdot \phi\|_{L^2(\Omega)} + \|\nabla \times \phi\|_{L^2(\Omega)}.$$

We now state some preparatory theorems.

Theorem 1 (surjectivity of curl) *A vector field $v \in L^2(\Omega)^3$ satisfies*

$$\nabla \cdot v = 0, \quad \langle v \cdot n, 1 \rangle_{\Gamma_i} = 0 \quad \text{for } 0 \leq i \leq p \quad (7)$$

if and only if there exists a vector potential ϕ in $H^1(\Omega)^3$ such that

$$v = \nabla \times \phi.$$

Furthermore,

$$\nabla \cdot \phi = 0.$$

Proof We outline a sketch of the proof for the 'only if' part. Let $v \in L^2(\Omega)^3$ satisfy (7). The compatibility condition $\langle v \cdot n, 1 \rangle_{\Gamma_i} = 0$, $0 \leq i \leq p$ ensures that we can extend v to the whole \mathbb{R}^3 so that the extended function \tilde{v} belongs to $L^2(\mathbb{R}^3)^3$, is divergence-free, and has compact support. Then the Fourier transform of \tilde{v} , denoted by $\mathcal{F}\tilde{v}$, is holomorphic, since \tilde{v} has compact support. After taking the Fourier transform, the conditions $\nabla \cdot \tilde{v} = 0$, $\tilde{v} = \nabla \times \phi$ and $\nabla \cdot \phi = 0$ convert to three algebraic equations which uniquely determine $\mathcal{F}\phi$, and ϕ is constructed by the inverse Fourier transform. The fact that $\phi \in H^1(\Omega)^3$ is a consequence of $\mathcal{F}\tilde{v}$ being holomorphic. \square

Theorem 2 (bijectivity on U_0) *Suppose $v \in L^2(\Omega)^3$ satisfies*

$$\nabla \cdot v = 0, \quad \langle v \cdot n, 1 \rangle_{\Gamma_i} = 0 \quad \text{for } 0 \leq i \leq p.$$

Then there exists a unique ϕ in $H(\text{curl}; \Omega)$ such that

$$\nabla \times \phi = v, \quad \nabla \cdot \phi = 0, \quad \phi \cdot n = 0;$$

it is characterized as the unique solution of the boundary value problem:

$$\begin{cases} \phi \in H, \\ -\Delta \phi = \nabla \times v \quad \text{in } H^{-1}(\Omega)^3, \\ (\nabla \times \phi - v) \cdot n = 0 \quad \text{on } \Gamma. \end{cases}$$

Proposition 3 *Let Ω be a bounded, connected and Lipschitz-continuous open subset of \mathbb{R}^d . The inhomogeneous Neumann's problem*

$$(N) \begin{cases} -\Delta u = f \quad \text{in } \Omega, \\ (\nabla u, \nabla v) = -(\Delta u, v) + \langle g, v \rangle_{\Gamma} \quad \forall v \in H^1(\Omega) \end{cases}$$

with $f \in L^2(\Omega)$ and $g \in H^{1/2}(\Gamma)$ admits a unique solution $\bar{u} \in H^1(\Omega)/\mathbb{R}$. Moreover, for each function u in the equivalence class \bar{u} , we have

$$|u|_{H^1(\Omega)} \leq C(\|f\|_{L^2(\Omega)} + \|g\|_{H^{-1/2}(\Gamma)}),$$

where $|\cdot|_{H^1(\Omega)}$ denotes the seminorm on $H^1(\Omega)$, and the constant C only depends on Ω .

Proof One can prove that $\bar{v} \rightarrow |v|_{H^1(\Omega)}$ defines a norm equivalent to that of the quotient norm of the Hilbert space $H^1(\Omega)/\mathbb{R}$. Then the problem (N) can be equivalently phrased as

$$a(\bar{u}, \bar{v}) = l(\bar{v}) \quad \forall \bar{v} \in H^1(\Omega)/\mathbb{R},$$

with the continuous elliptic bilinear form

$$a(\bar{u}, \bar{v}) \equiv (\nabla u, \nabla v) \quad u \in \bar{u}, v \in \bar{v}$$

and the linear functional

$$l(\bar{v}) \equiv (f, v) + \langle g, v \rangle_{\Gamma} \quad v \in \bar{v}.$$

The existence of a unique solution for $u \in H^1(\Omega)/\mathbb{R}$ follows from the Lax-Milgram theorem, and the inequality follows from the operator norm $\|l\| \leq \|f\|_{L^2(\Omega)} + \|g\|_{H^{-1/2}(\Gamma)}$ and the ellipticity of $a(\cdot, \cdot)$. \square

Theorem 4 *Suppose Ω is bounded, Lipschitz-continuous and simply-connected. Then the mapping $\phi \rightarrow \nabla \times \phi$ is an isomorphism from the space U_0 onto the space:*

$$T = \{u \in L^2(\Omega)^3; \nabla \cdot u = 0, \quad \langle u \cdot n, 1 \rangle_{\Gamma_i} = 0, \quad 0 \leq i \leq p\};$$

and there exist two positive constants C_1 and C_2 such that

$$\|\phi\|_{L^2(\Omega)} \leq C_1 \|\nabla \times \phi\|_{L^2(\Omega)} \quad \forall \phi \in U_0 \quad (8)$$

$$\|\phi\|_{L^2(\Omega)} \leq C_2 \{\|\nabla \times \phi\|_{L^2(\Omega)} + \|\nabla \cdot \phi\|_{L^2(\Omega)}\} \quad \forall \phi \in U \quad (9)$$

Proof The key ingredient in establishing the inequality (8) is the inverse mapping theorem: the curl map $\nabla \times : U_0 \rightarrow T$ is a continuous bijective linear operator from the Banach space U_0 onto the Banach space T ; by the inverse mapping theorem it has a bounded inverse $(\nabla \times)^{-1} : T \rightarrow U_0$. The bijectivity of the curl map is established in Theorem 2, while the continuity automatically follows from the definition of the subspace norm on U_0 inherited from U .

Every vector field $\phi \in L^2(\Omega)^3$ has the decomposition $\phi = \nabla q + \nabla \times \psi$, where $q \in H^1(\Omega)/\mathbb{R}$ is the only solution of

$$(\nabla q, \nabla \mu) = (\phi, \nabla \mu) \quad \forall \mu \in H^1(\Omega).$$

If $\phi \in U$, then $\nabla \psi \in U_0$, and applying Proposition 3 to ∇q yields $\|\nabla q\|_{L^2(\Omega)} = |q|_{H^1(\Omega)} \leq C \|\Delta q\|_{L^2(\Omega)} = C \|\nabla \cdot \phi\|_{L^2(\Omega)}$. Then (9) follows from (8) and the triangle inequality. \square

We are now in the position to prove our main result.

4 Proof of the main results

Since D is a $d \times d$ symmetric positive definite matrix, the eigenvalues of D are strictly positive real numbers. Then

$$\dot{\Sigma}_s = \int_{\Omega} F_{\text{irr}}(x) \cdot D^{-1} F_{\text{irr}}(x) \rho_{\text{inv}}(x) dx \leq \lambda^{-1} \int_{\Omega} |F_{\text{irr}}(x)|^2 \rho_{\text{inv}}(x) dx$$

where λ denotes the smallest eigenvalue of D . For convenience we shall assume $\lambda = 1$, hence to control $\dot{\Sigma}_s$ from above it suffices to find upper bounds for the quantity

$$\int_{\Omega} |F_{\text{irr}}(x)|^2 \rho_{\text{inv}}(x) dx = \int_{\Omega} |J_{\text{inv}}(x)|^2 \rho_{\text{inv}}^{-1}(x) dx$$

subject to the conditions $\nabla \cdot (J_{\text{inv}}) = 0$ in Ω and $J_{\text{inv}} \cdot n = 0$ on Γ .

Proposition 5 *Suppose the domain Ω is bounded, Lipschitz-continuous and simply-connected. Then there exists a positive constant $C(\Omega)$, depending only on Ω , such that*

$$\int_{\Omega} |J_{\text{inv}}|^2 dx \leq C(\Omega) \int_{\Omega} |\nabla \times J_{\text{inv}}|^2 dx$$

holds for any steady-state current J_{inv} in Ω .

Proof This follows directly from the inequality (8). □

Theorem 6 *Suppose the domain Ω is bounded, Lipschitz-continuous and simply-connected. Then there exists a positive constant $C(\Omega)$, depending only on Ω , such that*

$$\dot{\Sigma}_s \leq C(\Omega) \left(V + \frac{1}{4} \int_{\Omega} (|\nabla \Phi \times F_{\text{irr}}|^2 + |\nabla \Phi \cdot F_{\text{irr}}|^2) \rho_{\text{inv}} dx \right)$$

holds at the non-equilibrium steady state of any diffusion process in Ω .

Proof Since $J_{\text{inv}} \cdot n = (F_{\text{irr}} \rho_{\text{inv}}) \cdot n = 0$ on Γ , it follows that $(F_{\text{irr}} \rho_{\text{inv}}^{1/2}) \cdot n = 0$ on Γ . Applying Theorem 4 gives

$$\begin{aligned} \int_{\Omega} |F_{\text{irr}} \rho_{\text{inv}}^{1/2}|^2 dx &\leq C_1 \left(\int_{\Omega} |\nabla \times (F_{\text{irr}} \rho_{\text{inv}}^{1/2})|^2 dx + \int_{\Omega} |\nabla \cdot (F_{\text{irr}} \rho_{\text{inv}}^{1/2})|^2 dx \right) \\ &= C_1 \left(\int_{\Omega} |\rho_{\text{inv}}^{1/2} \nabla \times F_{\text{irr}} + \nabla \rho_{\text{inv}}^{1/2} \times F_{\text{irr}}|^2 dx \right. \\ &\quad \left. + \int_{\Omega} |\rho_{\text{inv}}^{1/2} \nabla \cdot F_{\text{irr}} + \nabla \rho_{\text{inv}}^{1/2} \cdot F_{\text{irr}}|^2 dx \right) \\ &\leq C_2 \int_{\Omega} \left(|\nabla \times F_{\text{irr}}|^2 + \frac{1}{4} |\nabla \Phi \times F_{\text{irr}}|^2 + \frac{1}{4} |\nabla \Phi \cdot F_{\text{irr}}|^2 \right) \rho_{\text{inv}} dx, \end{aligned}$$

where to obtain the last inequality we have used

$$\nabla \rho_{\text{inv}}^{1/2} = \frac{1}{2} \rho_{\text{inv}}^{-1/2} \nabla \rho_{\text{inv}} = -\frac{1}{2} \rho_{\text{inv}}^{1/2} \nabla \Phi$$

and

$$\nabla \cdot (F_{\text{irr}} \rho_{\text{inv}}) = \rho_{\text{inv}} \nabla \cdot F_{\text{irr}} + \nabla \rho_{\text{inv}} \cdot F_{\text{irr}} = 0 \implies \nabla \cdot F_{\text{irr}} = \nabla \Phi \cdot F_{\text{irr}}.$$

□

Corollary 7 *(the Freidlin-Wentzel limit) Suppose $\epsilon > 0$, consider the steady state of the diffusion process in Ω*

$$dx_{\tau} = F(x_{\tau})d\tau + \sqrt{2\epsilon} \hat{D} dW_{\tau},$$

where we use superscripts to emphasize the quantities' dependence on ϵ , and we introduced $\hat{D} \equiv D/\|D\|$. Then, as ϵ approaches zero, at the leading order in ϵ we have

$$\dot{\Sigma}_s^{\epsilon} \leq C(\Omega) \int_{\Omega} |\nabla \Phi^{\epsilon} \times F_{\text{irr}}|^2 \rho_{\text{inv}}^{\epsilon} dx \stackrel{D=\text{const} \mathbb{1}}{=} C(\Omega) \int_{\Omega} |\nabla \Phi^{\epsilon} \times F|^2 \rho_{\text{inv}}^{\epsilon} dx,$$

where $C(\Omega)$ is a constant depending only on the domain.

Proof According to the Large Deviation Theory [42] the limit

$$\lim_{\epsilon \rightarrow 0} \epsilon \ln \rho_{\text{inv}}^\epsilon(x) =: -\psi_0(x)$$

exists under the assumed confining conditions on F . Since $\Phi^\epsilon(x) \equiv -\ln \rho_{\text{inv}}^\epsilon(x)$, we write out the asymptotic expansion for $\Phi^\epsilon(x)$ as

$$\Phi^\epsilon(x) = \epsilon^{-1} \psi_0(x) + \psi_1(x) + O(\epsilon).$$

The decomposition (4) further gives the following representation formula for F

$$F(x) = -A^\epsilon(x) \nabla \Phi^\epsilon(x) + \nabla \cdot A^\epsilon(x) - \epsilon \hat{D} \nabla \Phi^\epsilon(x). \quad (10)$$

Since $\nabla \Phi^\epsilon(x)$ is of order ϵ^{-1} , Eq. (10) suggests that $\nabla \cdot A^\epsilon(x)$ is of order ϵ . Therefore, both $\nabla \times F_{\text{irr}}^\epsilon(x)$ and $\nabla \Phi^\epsilon(x) \cdot F_{\text{irr}}^\epsilon(x) = \nabla \Phi^\epsilon(x) \cdot (\nabla \cdot A^\epsilon(x))$ are of order 1, while $\nabla \Phi^\epsilon(x) \times F_{\text{irr}}^\epsilon(x)$ is of order ϵ^{-1} . Further, we have

$$\nabla \Phi^\epsilon(x) \times F(x) = \nabla \Phi^\epsilon(x) \times F_{\text{irr}}(x) - \epsilon \nabla \Phi^\epsilon(x) \times \hat{D} \nabla \Phi^\epsilon(x)$$

and if $D = \text{const} \cdot \mathbb{1}$ then also $\nabla \Phi^\epsilon(x) \times F(x) = \nabla \Phi^\epsilon(x) \times F_{\text{irr}}(x)$. Thus, to leading order in ϵ only contributions $\perp \nabla \Phi$ are important. The conclusion follows from Theorem 6. \square

Theorem 8 *Suppose the domain Ω is bounded, Lipschitz-continuous and simply-connected, and the steady state density ρ_{inv} satisfies*

$$\rho_{\text{inv}}|_\Gamma > 0; \quad \sup \rho_{\text{inv}} < \infty; \quad \rho_{\text{inv}}^{1/(1-q)} \in L^1(\Omega), \quad 1 < q < \infty.$$

Then there exists a constant $C(\Omega, \rho_{\text{inv}}) > 0$, depending on Ω and ρ_{inv} , such that

$$\dot{\Sigma}_s \leq C(\Omega, \rho_{\text{inv}}) \left(\left(\int_\Omega |\nabla \times F_{\text{irr}}|^{2q} \rho_{\text{inv}} dx \right)^{1/q} + \left(\int_\Omega |\nabla \Phi \cdot F_{\text{irr}}|^{2q} \rho_{\text{inv}} dx \right)^{1/q} \right)$$

and the constant $C(\Omega, \rho_{\text{inv}})$ takes the form

$$C(\Omega, \rho_{\text{inv}}) = C(\Omega) \cdot \sup_\Omega \rho_{\text{inv}} \cdot \left(\int_\Omega \rho_{\text{inv}}^{1/(1-q)} dx \right)^{(q-1)/q}.$$

Proof We have

$$\int_\Omega |F_{\text{irr}}|^2 \rho_{\text{inv}} dx \leq \sup_\Omega \rho_{\text{inv}} \int_\Omega |F_{\text{irr}}|^2 dx.$$

The assumption $\rho_{\text{inv}} > 0$ on Γ implies $F_{\text{irr}} \cdot n|_\Gamma = 0$, so we apply Theorem 4 to F_{irr} to get

$$\begin{aligned} \int_\Omega |F_{\text{irr}}|^2 dx &\leq C(\Omega) \left(\int_\Omega |\nabla \times F_{\text{irr}}|^2 dx + \int_\Omega |\nabla \cdot F_{\text{irr}}|^2 dx \right) \\ &= C(\Omega) \left(\int_\Omega |\nabla \times F_{\text{irr}}|^2 \rho_{\text{inv}}^{1/q} \rho_{\text{inv}}^{-1/q} dx + \int_\Omega |\nabla \Phi \cdot F_{\text{irr}}|^2 \rho_{\text{inv}}^{1/q} \rho_{\text{inv}}^{-1/q} dx \right) \\ &\leq C(\Omega) \left(\int_\Omega \rho_{\text{inv}}^{1/(1-q)} dx \right)^{(q-1)/q} \\ &\quad \left(\left(\int_\Omega |\nabla \times F_{\text{irr}}|^{2q} \rho_{\text{inv}} dx \right)^{1/q} + \left(\int_\Omega |\nabla \Phi \cdot F_{\text{irr}}|^{2q} \rho_{\text{inv}} dx \right)^{1/q} \right), \end{aligned}$$

where the constant $C(\Omega) > 0$ only depends on Ω , and in the last step we have used Hölder's inequality. \square

Remark 1 Since $\rho_{\text{inv}}^{1/(1-q)} \in L^1(\Omega)$ if and only if $\rho_{\text{inv}}^{-1} \in L^{1/(q-1)}(\Omega)$, to have q approach 1 we require higher integrability of ρ_{inv}^{-1} . If

$$\rho_{\text{inv}}^{-1} \in L^\infty(\Omega) \quad \text{i.e.} \quad \inf_{\Omega} \rho_{\text{inv}} > 0,$$

we can set $q = 1$ to get

$$\dot{\Sigma}_s \leq C(\Omega, \rho_{\text{inv}}) \left(V + \int_{\Omega} |\nabla \Phi \cdot F_{\text{irr}}|^2 \rho_{\text{inv}} dx \right),$$

where the constant $C(\Omega, \rho_{\text{inv}})$ takes the form

$$C(\Omega, \rho_{\text{inv}}) = C(\Omega) \cdot \sup_{\Omega} \rho_{\text{inv}} \cdot \left(\inf_{\Omega} \rho_{\text{inv}} \right)^{-1},$$

and in such cases ρ_{inv} defines a measure on Ω which is equivalent to the Lebesgue measure.

Corollary 9 *Suppose the domain Ω is bounded, Lipschitz-continuous and simply-connected, and the steady state density ρ_{inv} is strictly positive on Γ . Then there exists a constant $C(\Omega) > 0$ depending only on Ω , such that*

$$\dot{\Sigma}_s \leq C(\Omega) \cdot \sup_{\Omega} \rho_{\text{inv}} \left(\int_{\Omega} |\nabla \times F_{\text{irr}}|^2 dx + \int_{\Omega} |\nabla \Phi \cdot F_{\text{irr}}|^2 dx \right)$$

and

$$\dot{\Sigma}_s \leq C(\Omega) \cdot \sup_{\Omega} \rho_{\text{inv}} \cdot \left(\inf_{\Omega} \rho_{\text{inv}} \right)^{-1} \left(V + \int_{\Omega} |\nabla \Phi \cdot F_{\text{irr}}|^2 \rho_{\text{inv}} dx \right).$$

Proof The statements follow from the lines of the proof of Theorem 8. If $\inf_{\Omega} \rho_{\text{inv}} = 0$ we interpret the right-hand-side as infinity. \square

We have shown in Corollary 7 that when the steady state density is peaked, the entropy production rate is dominated by the vorticity-like quantity $\int_{\Omega} |\nabla \Phi \times F|^2 \rho_{\text{inv}} dx$. Conversely, if the potential $\Phi(x)$ is a parabola such that $|\nabla \Phi(x)|$ is small near the minimum of $\Phi(x)$ (and thus where the mass of ρ_{inv} is concentrated), e.g. ρ_{inv} is a Gaussian centered at the origin and $\nabla \Phi(x) = cx$ for some constant c , then the upper-bounds in Corollary 9 are also dominated by vorticity terms.

Corollary 10 *Suppose $\omega \subset \Omega$, $\Phi(x)$ is a parabola on ω with principal axes $\partial_{x_i}^2 \Phi(x) = K_i > 0$, $1 \leq i \leq d$, $\det \text{Hess}(\Phi) \geq 0$ and ω contains a local minimum of Φ . For $d = 2$, if the diameter of ω is much smaller than $(\Delta \Phi)^{-1/2}$ then $\int_{\omega} |\nabla \times F_{\text{irr}}|^2 dx$ dominates over $\int_{\omega} |F_{\text{irr}} \cdot \nabla \Phi|^2 dx$ in the sense that*

$$\int_{\omega} |F_{\text{irr}} \cdot \nabla \Phi|^2 dx \lesssim (\Delta \Phi) \text{diam}(\omega)^2 \int_{\omega} |\nabla \times F_{\text{irr}}|^2 dx.$$

For $d = 3$, we have the same conclusion provided that the system possesses certain symmetry, which will be specified in the proof.

Proof We first deal with the 2-dimensional case as the notation is simpler. We use (x, y) to denote the two independent variables and use the decomposition (4) to write

$$F_{\text{irr}} = (-a\partial_y\Phi, a\partial_x\Phi)^\top + (\partial_y a, -\partial_x a)^\top,$$

where $A = \begin{pmatrix} 0 & a(x, y) \\ -a(x, y) & 0 \end{pmatrix}$ and $F_{\text{irr}} = -A\nabla\Phi + \nabla \cdot A$. Then

$$F_{\text{irr}} \cdot \nabla\Phi = -\{\partial_y\Phi\}\{\partial_x a\} + \{\partial_x\Phi\}\{\partial_y a\}$$

where $\{\cdot\}$ signifies that the differentials only act within the bracket and

$$\nabla \times F_{\text{irr}} = \{\partial_y a\}\{\partial_y\Phi\} + \{\partial_x a\}\{\partial_x\Phi\} - \Delta a + a\Delta\Phi.$$

Since $\Delta\Phi = \sum_{i=1}^d K_i \equiv K$ is a positive constant, we can prove that $\|(-\Delta + K)a(x, y)\|_{L^2(\omega)}$ bounds $|a|_{H^1(\omega)}$ from above using the Fourier transform \mathcal{F} :

$$|\mathcal{F}(-\Delta + K)a| = (K + \xi_1^2 + \xi_2^2)|\mathcal{F}a| \geq \sqrt{K}(|\xi_1| + |\xi_2|)|\mathcal{F}a| = \sqrt{K}(|\mathcal{F}\partial_x a| + |\mathcal{F}\partial_y a|)$$

together with the Parseval's identity, which asserts that the Fourier transform on $L^2(\mathbb{R}^d)$ preserves the L^2 norm. If ω contains a local minimum of Φ , say at $(x_0, y_0) \in \omega$, then $\nabla\Phi(x_0, y_0) = 0$ and

$$\begin{aligned} |\partial_x\Phi(x, y)| &= \left| \int_0^1 \partial_x^2\Phi(x_0 + s(x-x_0), y_0 + s(y-y_0))(x-x_0) \right. \\ &\quad \left. + \partial_{xy}^2\Phi(x_0 + s(x-x_0), y_0 + s(y-y_0))(y-y_0) ds \right| \leq K \text{diam}(\omega). \end{aligned}$$

where the mixed derivative of Φ is controlled by K because $\det\text{Hess}(\Phi) \geq 0$. The same uniform bound also holds for $|\partial_y\Phi(x, y)|$. Putting everything together and setting $|\Phi|_1 \equiv \sup_\omega \{|\partial_x\Phi(x, y)|, |\partial_y\Phi(x, y)|\}$, we have

$$\begin{aligned} \int_\omega |F_{\text{irr}} \cdot \nabla\Phi|^2 dx &\lesssim |\Phi|_1^2 |a|_{H^1(\omega)}^2 \\ &\lesssim K^{-1} |\Phi|_1^2 \|(-\Delta + K)a(x, y)\|_{L^2(\omega)}^2 \\ &\lesssim K^{-1} |\Phi|_1^2 \int_\omega |\nabla \times F_{\text{irr}}|^2 dx \\ &\lesssim K \text{diam}(\omega)^2 \int_\omega |\nabla \times F_{\text{irr}}|^2 dx. \end{aligned}$$

Therefore, if the diameter of the subdomain ω is much smaller than $1/\sqrt{K}$, the vorticity is dominating over the level-set variations.

Now we consider the 3-dimensional case and use $(x, y, z) \in \mathbb{R}^3$ to denote the independent variables. Let

$$A = \begin{pmatrix} 0 & a_1(x, y, z) & a_2(x, y, z) \\ -a_1(x, y, z) & 0 & a_3(x, y, z) \\ -a_2(x, y, z) & -a_3(x, y, z) & 0 \end{pmatrix}$$

and set $F_{\text{irr}} = -A\nabla\Phi + \nabla \cdot A$. We compute

$$\begin{aligned} F_{\text{irr}} \cdot \nabla\Phi &= \{\partial_z a_3\}\{\partial_y\Phi\} - \{\partial_x a_1\}\{\partial_y\Phi\} - \{\partial_y a_3\}\{\partial_z\Phi\} - \{\partial_x a_2\}\{\partial_z\Phi\} \\ &\quad + \{\partial_z a_2\}\{\partial_x\Phi\} + \{\partial_y a_1\}\{\partial_x\Phi\}, \end{aligned}$$

which involves only first order derivatives of a_i and Φ . Moreover, we have

$$\begin{aligned} (\nabla \times F_{\text{irr}})_1 &= -\partial_z^2 a_3 + a_3 \partial_z^2 \Phi - \partial_y^2 a_3 + a_3 \partial_y^2 \Phi \\ &\quad + \partial_{xz}^2 a_1 - a_1 \partial_{xz}^2 \Phi - \partial_{xy}^2 a_2 + a_2 \partial_{xy}^2 \Phi \\ &\quad + \{\partial_z a_3\}\{\partial_z\Phi\} + \{\partial_y a_3\}\{\partial_y\Phi\} - \{\partial_z a_1\}\{\partial_x\Phi\} + \{\partial_y a_2\}\{\partial_x\Phi\}, \end{aligned}$$

and similar expressions for the other two components. Now the difficulty arises because the second-order differential operators on a'_i s are no longer elliptic, and we require certain symmetry conditions to control the mixed derivatives as well as the first order terms. For instance, the assertion of the theorem holds if the three spatial dimensions are independent and self-similar, meaning that $\partial_{x_i x_j}^2 \Phi = 0$, $\forall i \neq j$, $\partial_{x_i x_j}^2 a_k = 0$, $\forall i \neq j$, $1 \leq k \leq 3$ and $\{\partial_{x_i} a_j : 1 \leq i, j \leq 3\}$ are multiples of one another. Then $\|\nabla \times F_{\text{irr}}\|_{L^2(\omega)}$ bounds $\|F_{\text{irr}} \cdot \nabla \Phi\|_{L^2(\omega)}$ from above, and it is dominating when the diameter of ω is small relative to $1/\sqrt{K}$. \square

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